

A Liouville comparison principle for sub- and super-solutions of the equation

$$w_t - \Delta_p(w) = |w|^{q-1}w.$$

Vasilii V. Kurta.

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Abstract

We establish a Liouville comparison principle for entire weak sub- and super-solutions of the equation $(*)$ $w_t - \Delta_p(w) = |w|^{q-1}w$ in the half-space $\mathbb{S} = \mathbb{R}_+^1 \times \mathbb{R}^n$, where $n \geq 1$, $q > 0$ and $\Delta_p(w) := \operatorname{div}_x(|\nabla_x w|^{p-2} \nabla_x w)$, $1 < p \leq 2$. In our study we impose neither restrictions on the behaviour of entire weak sub- and super-solutions on the hyper-plane $t = 0$, nor any growth conditions on their behaviour and on that of any of their partial derivatives at infinity. We prove that if $1 < q \leq p - 1 + \frac{p}{n}$, and u and v are, respectively, an entire weak super-solution and an entire weak sub-solution of $(*)$ in \mathbb{S} which belong, only locally in \mathbb{S} , to the corresponding Sobolev space and are such that $u \geq v$, then $u \equiv v$. The result is sharp. As direct corollaries we obtain known Fujita-type and Liouville-type theorems.

1 Introduction and definitions.

The purpose of this work is to obtain a Liouville comparison principle of elliptic type for entire weak sub- and super-solutions of the equation

$$w_t - \Delta_p(w) = |w|^{q-1}w \tag{1}$$

in the half-space $\mathbb{S} = (0, +\infty) \times \mathbb{R}^n$, where $n \geq 1$ is a natural number, $q > 0$ is a real number and $\Delta_p(w) := \sum_{i=1}^n \frac{d}{dx_i} A_i(\nabla w)$, with $A_i(\xi) = |\xi|^{p-2} \xi_i$ for all

$\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $p > 1$, defines the well-known p -Laplacian operator. Under entire sub- and super-solutions of (1) we understand sub- and super-solutions of (1) defined in the whole half-space \mathbb{S} , and under Liouville results of elliptic type for sub- and super-solutions of the parabolic equation (1) in the half-space \mathbb{S} we understand Liouville-type results which, in their formulations, have no restrictions on the behaviour of sub- and super-solutions of (1) on the hyper-plane $t = 0$. Also, we would like to underline that we impose neither growth conditions on the behaviour of sub- and super-solutions to (1) or on that of any of their partial derivatives at infinity.

Definition 1 *Let $n \geq 1$, $p > 1$ and $q > 0$. A function $u = u(t, x)$ defined and measurable in \mathbb{S} is called an entire weak super-solution of the equation (1) in \mathbb{S} if it belongs to the function space $L_{q,\text{loc}}(\mathbb{S})$, with $u_t \in L_{1,\text{loc}}(\mathbb{S})$ and $|\nabla_x u|^p \in L_{1,\text{loc}}(\mathbb{S})$, and satisfies the integral inequality*

$$\int_{\mathbb{S}} \left[u_t \varphi + \sum_{i=1}^n |\nabla_x u|^{p-2} u_{x_i} \varphi_{x_i} - |u|^{q-1} u \varphi \right] dt dx \geq 0 \quad (2)$$

for every non-negative function $\varphi \in C^\infty(\mathbb{S})$ with compact support in \mathbb{S} , where $C^\infty(\mathbb{S})$ is the space of all functions defined and infinitely differentiable in \mathbb{S} .

Definition 2 *A function $v = v(t, x)$ is an entire weak sub-solution of (1) if $u = -v$ is an entire weak super-solution of (1).*

2 Results.

Theorem 1 *Let $n \geq 1$, $2 \leq p > 1$ and $1 < q \leq p - 1 + \frac{p}{n}$, and let u be an entire weak super-solution and v an entire weak sub-solution of (1) in \mathbb{S} such that $u \geq v$. Then $u = v$ in \mathbb{S} .*

The result in Theorem 1, which evidently has a comparison principle character, we term a Liouville-type comparison principle, since, in the particular cases when $u \equiv 0$ or $v \equiv 0$, it becomes a Liouville-type theorem of elliptic type, respectively, for entire weak sub-solutions or entire weak super-solutions of (1).

Since in Theorem 1 we impose no conditions on the behaviour of entire weak sub- or super-solutions of the equation (1) on the hyper-plane $t = 0$, we

can formulate, as a direct corollary of Theorem 1, the following comparison principle, which in turn one can term a Fujita comparison principle, for entire weak sub- and super-solutions of the Cauchy problem for the equation (1). It is clear that in the particular cases when $u \equiv 0$ or $v \equiv 0$, it becomes a Fujita-type theorem, respectively, for entire weak sub-solutions or entire weak super-solutions of the Cauchy problem for the equation (1).

Theorem 2 *Let $n \geq 1$, $2 \geq p > 1$ and $1 < q \leq p - 1 + \frac{p}{n}$, and let u be an entire weak super-solution and v an entire weak sub-solution of the Cauchy problem, with possibly different initial data for u and v , for the equation (1) in \mathbb{S} such that $u \geq v$. Then $u = v$ in \mathbb{S} .*

Remark 1 *The initial data for u and v in Theorem 2 may be different.*

Note that the results in Theorems 1 and 2 are sharp and that the hypotheses on the parameter p in these theorems in fact force p to be greater than $\frac{2n}{n+1}$. The sharpness of these results for $q > p - 1 + \frac{p}{n} \geq 1$ follows, for example, from the existence of non-negative self-similar entire solutions to (1) in \mathbb{S} , which was shown in [1]. Also, there one can find a Fujita-type theorem on the non-existence of non-negative entire solutions of the Cauchy problem for (1), which was obtained as a very interesting generalization of the famous blow-up result established in [4], [5] and [9] to quasilinear parabolic equations. For $0 < q \leq 1$, it is evident that the function $u(t, x) = e^t$ is a positive entire classical super-solution of (1) in \mathbb{S} .

We would also like to note that the results of the present work were announced in [13] and that similar results for solutions of semilinear parabolic inequalities were obtained in [7]. To prove the results we use the α -monotonicity property of the p -Laplacian operator which was established in [11] and continue to develop an approach in [7] and [8], the elliptic analogue of which was proposed [11]. That approach was subsequently used and developed in the same framework by E. Mitidieri, S. Pokhozhaev and many others, almost none of which cite the original research in [11].

For a survey of the literature on the asymptotic behaviour of and blow-up results for solutions, sub- and super-solutions of the Cauchy problem for nonlinear parabolic equations we refer to [2], [3], [6], [14], [15] and [16].

3 Proofs.

In what follows, for $q > 1$ and $2 \geq p > 1$, let

$$\omega = \frac{p(q-1)}{q-p+1} \quad (3)$$

and

$$P(R) = \{(t, x) \in \mathbb{S} : t^{2/\omega} + |x|^2 < R^{2/\omega}\}$$

for all $R > 0$. In this case it is clear that $0 < \omega \leq 2$ and that the inequality

$$\text{volume of } P(R) \leq cR^{\frac{n+\omega}{\omega}}, \quad (4)$$

with c some positive constant which depends possibly only on n and ω , holds for all $R > 0$.

Proof of Theorem 1. Let $n \geq 1$, $2 \geq p > 1$ and $1 < q \leq p - 1 + \frac{p}{n}$, and let u be an entire weak super-solution and v an entire weak sub-solution of (1) in \mathbb{S} such that $u \geq v$. By the well-known inequality

$$(|u|^{q-1}u - |v|^{q-1}v)(u - v) \geq 2^{1-q}|u - v|^{q+1}$$

which holds for every $q \geq 1$ and all $u, v \in \mathbb{R}^1$ we obtain from (2) the relation

$$\begin{aligned} \int_{\mathbb{S}} \left[(u - v)_t \varphi + \sum_{i=1}^n \varphi_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) \right] dt dx \geq \\ 2^{1-q} \int_{\mathbb{S}} (u - v)^q \varphi dt dx, \end{aligned} \quad (5)$$

which holds for every non-negative function $\varphi \in C^\infty(\mathbb{S})$ with compact support in \mathbb{S} . Let $\tau > 0$ and $R > 0$ be real numbers. Let $\eta : [0, +\infty) \rightarrow [0, 1]$ be a C^∞ -function which has the non-negative derivative η' and equals 0 on the interval $[0, \tau]$ and 1 on the interval $[2\tau, +\infty)$, and let $\zeta : [0, +\infty) \times \mathbb{R}^n \rightarrow [0, 1]$ be a C^∞ -function which equals 1 on $\overline{P(R/2)}$ and 0 on $\{[0, +\infty) \times \mathbb{R}^n\} \setminus \overline{P(R)}$. Let $\varphi(t, x) = (w(t, x) + \varepsilon)^{-\nu} \zeta^s(t, x) \eta^2(t)$, where $w(t, x) = u(t, x) - v(t, x)$, $\varepsilon > 0$ and the positive constants $s > 1$ and $\nu \in (0, p - 1)$ will be chosen

below. Substituting the function φ in (5) and then integrating by parts we arrive at

$$\begin{aligned}
& -\frac{s}{1-\nu} \int_{P(R)} (w+\varepsilon)^{1-\nu} \zeta_t \zeta^{s-1} \eta^2 dt dx - \frac{2}{1-\nu} \int_{P(R)} (w+\varepsilon)^{1-\nu} \zeta^s \eta' \eta dt dx \\
& -\nu \int_{P(R)} \sum_{i=1}^n w_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) (w+\varepsilon)^{-\nu-1} \zeta^s \eta^2 dt dx \\
& +s \int_{P(R)} \sum_{i=1}^n \zeta_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) (w+\varepsilon)^{-\nu} \zeta^{s-1} \eta^2 dt dx \\
& \equiv I_1 + I_2 + I_3 + I_4 \geq 2^{1-q} \int_{P(R)} w^q (w+\varepsilon)^{-\nu} \zeta^s \eta^2 dt dx. \quad (6)
\end{aligned}$$

First, observing that I_3 is non-positive, we estimate I_4 in terms of I_3 using the fact, which is a key point in our proof, that for $1 < p \leq 2$ the p -Laplacian operator Δ_p satisfies the α -monotonicity condition (see, e.g., [12]) with $\alpha = p$. This in our case consists mostly of the fact that there exists a positive constant \mathcal{K} such that the coefficients A_i , $i = 1, \dots, n$, of the p -Laplacian operator satisfy the inequality

$$\left(\sum_{i=1}^n (A_i(\xi^1) - A_i(\xi^2))^2 \right)^{\alpha/2} \leq \mathcal{K} \left(\sum_{i=1}^n (\xi_i^1 - \xi_i^2) (A_i(\xi^1) - A_i(\xi^2)) \right)^{\alpha-1} \quad (7)$$

for all pairs $\xi^1, \xi^2 \in \mathbb{R}^n$ and $\alpha = p$, provided $1 < p \leq 2$. As a result, we have the relation

$$\begin{aligned}
& |I_4| \\
& \leq \int_{P(R)} c_1 |\nabla_x \zeta| \left(\sum_{i=1}^n w_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) \right)^{\frac{p-1}{p}} (w+\varepsilon)^{-\nu} \zeta^{s-1} \eta^2 dt dx. \quad (8)
\end{aligned}$$

Here we use the symbols c_i , $i = 1, \dots, 8$, to denote constants depending possibly on n , p , q , s or ν but not on R , ε or τ . Further, estimating the integrand on the right-hand side of (8) by Young's inequality

$$\mathcal{A}\mathcal{B} \leq \rho \mathcal{A}^{\frac{\beta}{\beta-1}} + \rho^{1-\beta} \mathcal{B}^\beta \quad (9)$$

with $\rho = \frac{\nu}{2}, \beta = p$,

$$\mathcal{A} = \left(\sum_{i=1}^n w_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) \right)^{\frac{p-1}{p}} (w + \varepsilon)^{\frac{(1+\nu)(1-p)}{p}} \zeta^{\frac{s(p-1)}{p}} \eta^{\frac{2(p-1)}{p}}$$

and

$$\mathcal{B} = c_1 |\nabla_x \zeta| (w + \varepsilon)^{\frac{p-1-\nu}{p}} \zeta^{\frac{s}{p}-1} \eta^{\frac{2}{p}},$$

we arrive at

$$\begin{aligned} |I_4| \leq & \frac{\nu}{2} \int_{P(R)} \sum_{i=1}^n w_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) (w + \varepsilon)^{-\nu-1} \zeta^s \eta^2 dt dx \\ & + \int_{P(R)} c_2 (w + \varepsilon)^{p-1-\nu} |\nabla_x \zeta|^p \zeta^{s-p} \eta^2 dt dx. \end{aligned} \quad (10)$$

Now, observing that I_2 in (6) is also non-positive, we obtain from (6) and (10) the relation

$$\begin{aligned} & \int_{P(R)} c_2 (w + \varepsilon)^{1-\nu} |\zeta_t| \zeta^{s-1} \eta^2 dt dx + \int_{P(R)} c_2 (w + \varepsilon)^{p-1-\nu} |\nabla_x \zeta|^p \zeta^{s-p} \eta^2 dt dx \\ & \geq \int_{P(R)} w^q (w + \varepsilon)^{-\nu} \zeta^s \eta^2 dt dx \\ & + \int_{P(R)} \sum_{i=1}^n w_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) (w + \varepsilon)^{-\nu-1} \zeta^s \eta^2 dt dx. \end{aligned} \quad (11)$$

Estimating both integrands on the left-hand side of (11) by Young's inequality (9) with $\rho = \frac{1}{2}, \beta = \frac{q-\nu}{q-1}$,

$$\mathcal{A} = (w + \varepsilon)^{1-\nu} \zeta^{\frac{s(1-\nu)}{q-\nu}} \eta^{\frac{2(1-\nu)}{q-\nu}},$$

$$\mathcal{B} = c_2 |\zeta_t| \zeta^{\frac{s(q-1)}{q-\nu}-1} \eta^{\frac{2(q-1)}{q-\nu}}$$

and $\rho = \frac{1}{2}, \beta = \frac{q-\nu}{q-p+1}$,

$$\mathcal{A} = (w + \varepsilon)^{p-1-\nu} \zeta^{\frac{s(p-1-\nu)}{q-\nu}} \eta^{\frac{2(p-1-\nu)}{q-\nu}},$$

$$\mathcal{B} = c_2 |\nabla_x \zeta|^p \zeta^{\frac{s(q-p+1)}{q-\nu}-p} \eta^{\frac{2(q-p+1)}{q-\nu}},$$

respectively, we have the relation

$$\begin{aligned} & \frac{1}{2} \int_{P(R)} (w + \varepsilon)^{q-\nu} \zeta^s \eta^2 dt dx + c_3 \int_{P(R)} |\zeta_t|^{\frac{q-\nu}{q-1}} \zeta^{s-\frac{q-\nu}{q-1}} \eta^2 dt dx \\ & + \frac{1}{2} \int_{P(R)} (w + \varepsilon)^{q-\nu} \zeta^s \eta^2 dt dx + c_3 \int_{P(R)} |\nabla_x \zeta|^{\frac{p(q-\nu)}{q-p+1}} \zeta^{s-\frac{p(q-\nu)}{q-p+1}} \eta^2 dt dx \\ & \geq \int_{P(R)} w^q (w + \varepsilon)^{-\nu} \zeta^s \eta^2 dt dx + \\ & \int_{P(R)} \sum_{i=1}^n w_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) (w + \varepsilon)^{-\nu-1} \zeta^s \eta^2 dt dx. \end{aligned} \quad (12)$$

Further, we estimate the integral

$$\int_{P(R)} w^q \zeta^s \eta^2 dt dx$$

by the inequality (12). To this end, we substitute

$$\varphi(t, x) = \zeta^s(t, x) \eta^2(t)$$

in (5) and after integration by parts there we obtain

$$\begin{aligned} & -s \int_{P(R)} w \zeta_t \zeta^{s-1} \eta^2 dt dx - 2 \int_{P(R)} w \zeta^s \eta' \eta dt dx \\ & + s \int_{P(R)} \sum_{i=1}^n \zeta_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) \zeta^{s-1} \eta^2 dt dx \\ & \geq 2^{1-q} \int_{P(R)} w^q \zeta^s \eta^2 dt dx. \end{aligned} \quad (13)$$

Since the second term on the left-hand side of (13) is non-positive we have

$$s \int_{P(R)} w |\zeta_t| \zeta^{s-1} \eta^2 dt dx$$

$$+s \int_{P(R)} \sum_{i=1}^n \zeta_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) \zeta^{s-1} \eta^2 dt dx \geq 2^{1-q} \int_{P(R)} w^q \zeta^s \eta^2 dt dx. \quad (14)$$

Now, estimating the first integral on the left-hand side of (14) by Hölder's inequality, we arrive at

$$\begin{aligned} & s \left(\int_{P(R) \setminus P(R/2)} w^q \zeta^s \eta^2 dt dx \right)^{\frac{1}{q}} \left(\int_{P(R)} |\zeta_t|^{\frac{q}{q-1}} \zeta^{s-\frac{q}{q-1}} \eta^2 dt dx \right)^{\frac{q-1}{q}} \\ & + s \int_{P(R)} \sum_{i=1}^n \zeta_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) \zeta^{s-1} \eta^2 dt dx \\ & \geq 2^{1-q} \int_{P(R)} w^q \zeta^s \eta^2 dt dx. \end{aligned} \quad (15)$$

On the other hand, by (7) we have

$$\begin{aligned} & \int_{P(R)} \sum_{i=1}^n \zeta_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) \zeta^{s-1} \eta^2 dt dx \\ & \leq c_4 \int_{P(R)} |\nabla_x \zeta| (w_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}))^{\frac{p-1}{p}} \zeta^{s-1} \eta^2 dt dx. \end{aligned} \quad (16)$$

Estimating the right-hand side of (16) by Hölder's inequality we arrive at the relation

$$\begin{aligned} & \int_{P(R)} \sum_{i=1}^n \zeta_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) \zeta^{s-1} \eta^2 dt dx \\ & \leq c_4 \left(\int_{P(R)} (w + \varepsilon)^{(1+\nu)(p-1)} |\nabla_x \zeta|^p \zeta^{s-p} \eta^2 dt dx \right)^{1/p} \\ & \times \left(\int_{P(R)} \sum_{i=1}^n w_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) (w + \varepsilon)^{-\nu-1} \zeta^s \eta^2 dt dx \right)^{\frac{p-1}{p}} \end{aligned} \quad (17)$$

which holds for every $\varepsilon > 0$ and $p - 1 > \nu > 0$. Further, for any $d > 1$ we have

$$\begin{aligned}
& \int_{P(R)} (w + \varepsilon)^{(1+\nu)(p-1)} |\nabla_x \zeta|^p \zeta^{s-p} \eta^2 dt dx \\
& \leq \left(\int_{P(R) \setminus P(R/2)} (w + \varepsilon)^{d(1+\nu)(p-1)} \zeta^s \eta^2 dt dx \right)^{\frac{1}{d}} \\
& \quad \times \left(\int_{P(R)} |\nabla_x \zeta|^{\frac{pd}{d-1}} \zeta^{s-\frac{pd}{d-1}} \eta^2 dt dx \right)^{\frac{d-1}{d}}. \tag{18}
\end{aligned}$$

Now, we choose for every $q > 1$ and a sufficiently small ν from the interval $(0, p-1)$ the parameter d such that $d(1+\nu)(p-1) = q$. Then (17) and (18) yield

$$\begin{aligned}
& \int_{P(R)} \sum_{i=1}^n \zeta_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) \zeta^{s-1} \eta^2 dt dx \\
& \leq c_4 \left(\int_{P(R) \setminus P(R/2)} (w + \varepsilon)^q \zeta^s \eta^2 dt dx \right)^{\frac{1}{pd}} \left(\int_{P(R)} |\nabla_x \zeta|^{\frac{pd}{d-1}} \zeta^{s-\frac{pd}{d-1}} \eta^2 dt dx \right)^{\frac{d-1}{pd}} \\
& \quad \times \left(\int_{P(R)} \sum_{i=1}^n w_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) (w + \varepsilon)^{-\nu-1} \zeta^s \eta^2 dt dx \right)^{\frac{p-1}{p}}. \tag{19}
\end{aligned}$$

Estimating the last term on the right-hand side of (19) by (12), we have

$$\begin{aligned}
& \int_{P(R)} \sum_{i=1}^n \zeta_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) \zeta^{s-1} \eta^2 dt dx \\
& \leq c_4 \left(\int_{P(R) \setminus P(R/2)} (w + \varepsilon)^q \zeta^s \eta^2 dt dx \right)^{\frac{1}{pd}} \left(\int_{P(R)} |\nabla_x \zeta|^{\frac{pd}{d-1}} \zeta^{s-\frac{pd}{d-1}} \eta^2 dt dx \right)^{\frac{d-1}{pd}}
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_{P(R)} (w + \varepsilon)^{q-\nu} \zeta^s \eta^2 dt dx - \int_{P(R)} w^q (w + \varepsilon)^{-\nu} \zeta^s \eta^2 dt dx \right. \\
& \left. + c_3 \int_{P(R)} |\zeta_t|^{\frac{q-\nu}{q-1}} \zeta^{s-\frac{q-\nu}{q-1}} \eta^2 dt dx + c_3 \int_{P(R)} |\nabla_x \zeta|^{\frac{p(q-\nu)}{q-p+1}} \zeta^{s-\frac{p(q-\nu)}{q-p+1}} \eta^2 dt dx \right)^{\frac{p-1}{p}}. \quad (20)
\end{aligned}$$

In (20), passing to the limit as $\varepsilon \rightarrow 0$ as justified by Lebesgue's theorem (see, e.g., [10, p. 303]) we obtain

$$\begin{aligned}
& \int_{P(R)} \sum_{i=1}^n \zeta_{x_i} (|\nabla_x u|^{p-2} u_{x_i} - |\nabla_x v|^{p-2} v_{x_i}) \zeta^{s-1} \eta^2 dt dx \\
& \leq c_5 \left(\int_{P(R) \setminus P(R/2)} w^q \zeta^s \eta^2 dt dx \right)^{\frac{1}{pd}} \left(\int_{P(R)} |\nabla_x \zeta|^{\frac{pd}{d-1}} \zeta^{s-\frac{pd}{d-1}} \eta^2 dt dx \right)^{\frac{d-1}{pd}} \\
& \times \left(\int_{P(R)} |\zeta_t|^{\frac{q-\nu}{q-1}} \zeta^{s-\frac{q-\nu}{q-1}} \eta^2 dt dx + \int_{P(R)} |\nabla_x \zeta|^{\frac{p(q-\nu)}{q-p+1}} \zeta^{s-\frac{p(q-\nu)}{q-p+1}} dt dx \right)^{\frac{p-1}{p}}. \quad (21)
\end{aligned}$$

Further, (15) and (21) yield

$$\begin{aligned}
& \int_{P(R)} w^q \zeta^s \eta^2 dt dx \leq c_6 \left(\int_{P(R) \setminus P(R/2)} w^q \zeta^s \eta^2 dt dx \right)^{\frac{1}{q}} \left(\int_{P(R)} |\zeta_t|^{\frac{q}{q-1}} \zeta^{s-\frac{q}{q-1}} \eta^2 dt dx \right)^{\frac{q-1}{q}} \\
& + c_6 \left(\int_{P(R) \setminus P(R/2)} w^q \zeta^s \eta^2 dt dx \right)^{\frac{1}{pd}} \left(\int_{P(R)} |\nabla_x \zeta|^{\frac{pd}{d-1}} \zeta^{s-\frac{pd}{d-1}} \eta^2 dt dx \right)^{\frac{d-1}{pd}} \\
& \times \left(\int_{P(R)} |\zeta_t|^{\frac{q-\nu}{q-1}} \zeta^{s-\frac{q-\nu}{q-1}} \eta^2 dt dx + \int_{P(R)} |\nabla_x \zeta|^{\frac{p(q-\nu)}{q-p+1}} \zeta^{s-\frac{p(q-\nu)}{q-p+1}} \eta^2 dt dx \right)^{\frac{p-1}{p}}. \quad (22)
\end{aligned}$$

Now, for arbitrary $(t, x) \in \mathbb{S}$ and $R > 0$, we choose in (22) the function

$\zeta = \zeta(t, x)$ in the form

$$\zeta(t, x) = \psi \left(\frac{t^{2/\omega} + |x|^2}{R^{2/\omega}} \right), \quad (23)$$

where $0 < \omega \leq 2$ is given by (3) and $\psi : [0, \infty) \rightarrow [0, 1]$ is a C^∞ -function which equals 1 on $[0, 2^{-\frac{2}{\omega}}]$ and 0 on $[1, \infty)$ and is such that the inequalities

$$|\zeta_t| \leq c_7 R^{-1} \quad \text{and} \quad |\nabla_x \zeta| \leq c_7 R^{-\frac{1}{\omega}} \quad (24)$$

hold. Note that it is always possible to find such a function ζ . Indeed, this can be easily verified by direct calculation of the corresponding derivatives of the function ζ given by (23). Also, choosing in (22) the parameter s sufficiently large, we have from (22) by (4) and (24) the relation

$$\begin{aligned} \int_{P(R)} w^q \zeta^s \eta^2 dt dx &\leq c_8 \left(R^{\frac{n+\omega}{\omega} - \frac{q}{q-1}} \right)^{\frac{q-1}{q}} \left(\int_{P(R) \setminus P(R/2)} w^q \zeta^s \eta^2 dt dx \right)^{\frac{1}{q}} \\ &+ c_8 \left(R^{\frac{n+\omega}{\omega} - \frac{pd}{\omega(d-1)}} \right)^{\frac{d-1}{pd}} \left(R^{\frac{n+\omega}{\omega} - \frac{q-\nu}{q-1}} + R^{\frac{n+\omega}{\omega} - \frac{p(q-\nu)}{\omega(q-p+1)}} \right)^{\frac{p-1}{p}} \left(\int_{P(R) \setminus P(R/2)} u^q \zeta^s \eta^2 dt dx \right)^{\frac{1}{pd}}, \end{aligned}$$

which in turn by (3) implies

$$\begin{aligned} \int_{P(R)} w^q \zeta^s \eta^2 dt dx &\leq c_8 \left(R^{\frac{n+\omega}{\omega} - \frac{q}{q-1}} \right)^{\frac{q-1}{q}} \left(\int_{P(R) \setminus P(R/2)} w^q \zeta^s \eta^2 dt dx \right)^{\frac{1}{q}} \\ &+ c_8 \left(R^{\frac{n+\omega}{\omega} - \frac{pd}{\omega(d-1)}} \right)^{\frac{d-1}{pd}} \left(R^{\frac{n+\omega}{\omega} - \frac{q-\nu}{q-1}} \right)^{\frac{p-1}{p}} \left(\int_{P(R) \setminus P(R/2)} u^q \zeta^s \eta^2 dt dx \right)^{\frac{1}{pd}}. \quad (25) \end{aligned}$$

Making simple calculation in (25) we arrive at

$$\int_{P(R)} w^q \zeta^s \eta^2 dt dx \leq c_8 R^{\frac{n}{pq} [q-p+1 - \frac{p}{n}]} \left(\int_{P(R) \setminus P(R/2)} w^q \zeta^s \eta^2 dt dx \right)^{\frac{1}{q}}$$

$$+c_8 R^{\frac{n(pq-p+1-\nu(p-1))}{p^2q(q-1)}[q-p+1-\frac{p}{n}]} \left(\int_{P(R) \setminus P(R/2)} w^q \zeta^s \eta^2 dt dx \right)^{\frac{1}{pd}}. \quad (26)$$

Further, since for $q > 1$, $2 \geq p > 1$ and $p-1 > \nu > 0$ the quantities

$$\frac{n}{pq} \quad \text{and} \quad \frac{n(pq-p+1-\nu(p-1))}{p^2q(q-1)}$$

are positive, we obtain from (26) for $1 < q < p-1 + \frac{p}{n}$ the relation

$$\int_{\mathbb{S}} u^q \eta^2 dt dx = 0. \quad (27)$$

Also, for $q = p-1 + \frac{p}{n}$ we deduce from (26) that

$$\int_{\mathbb{S}} u^q \eta^2 dt dx < \infty.$$

The latter yields the relation

$$\int_{P(R_k) \setminus P(R_k/2)} w^q \eta^2 dt dx \rightarrow 0 \quad (28)$$

which holds for every sequence $R_k \rightarrow \infty$. On the other hand, the inequality

$$\begin{aligned} \int_{P(R/2)} w^q \eta^2 dt dx &\leq c_8 R^{\frac{n}{pq}[q-p+1-\frac{p}{n}]} \left(\int_{P(R) \setminus P(R/2)} w^q \eta^2 dt dx \right)^{\frac{1}{q}} \\ &+ c_8 R^{\frac{n(pq-p+1-\nu(p-1))}{p^2q(q-1)}[q-p+1-\frac{p}{n}]} \left(\int_{P(R) \setminus P(R/2)} w^q \eta^2 dt dx \right)^{\frac{1}{pd}} \end{aligned} \quad (29)$$

follows easily from (26). In turn, (28) and (29) imply for $q = p-1 + \frac{p}{n}$ that the relation

$$\int_{P(R_k)} u^q \eta^2 dt dx \rightarrow 0$$

holds for every sequence $R_k \rightarrow \infty$. The latter implies that (27) holds for every q satisfying

$$1 < q \leq p - 1 + \frac{p}{n}. \quad (30)$$

In (27), by letting the parameter τ in the definition of the function η tend to zero, we obtain that $u(t, x) = v(t, x)$ a.e. in \mathbb{S} for every q which satisfies (30). \square

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Author’s address:

Vasilii V. Kurta
American Mathematical Society
Mathematical Reviews
416 Fourth Street, P.O. Box 8604
Ann Arbor, Michigan 48107-8604, USA
e-mail: vkurta@umich.edu, vvk@ams.org